

Analytical methods and Computational algorithms for Analyzing the Structures of the Distribution of Primitive Roots of Sets of Consecutive Prime Numbers

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Abstract

The distribution of the primitive roots of consecutive primes has been the subject of research of many mathematicians for decades and remains so today. Understanding the structural properties of the distributions of primitive roots can help to establish the fundamental nature of prime numbers and their connections with other mathematical concepts and concepts from the field of information technology, including information theory. An attempt was made to categorize prime numbers depending on their indices, and the distribution of these indices by prime modulo was investigated. A comparative analysis of analytical and computational methods, algorithms for constructing classes of prime numbers and estimating constants in the extended and generalized Artin's conjecture is given. Methods for studying the structures of the primitive roots of systems of primes in sets of consecutive primes and analyzing their influence on the processes of forming Artin's constants have been developed. A systematic analysis of the factors allowed us to find regularities that significantly affect the processes of forming classes of distribution of primitive roots. It is proved that the properties of the classes of distribution of primitive roots in systems of sets of consecutive prime numbers in sets of a certain power determine stable estimates of a range of their properties. In general, this article contains an exhaustive review of the principles of formation of primitive roots and their distribution, a study of the dependence of estimates of Artin's constants on the class of prime numbers, and highlights potential avenues for future research in this area.

Keywords—distribution of primitive roots, Artin's constants, classes of prime numbers, distribution of indices modulo prime, distances between primitive roots.

Introduction

The subject of the study will be the theory of the formation of systems of primitive roots of prime numbers, which from the algebraic point of view are the basic elements of groups Fp^* . The laws of distribution of primitive roots for any prime number p are not generally known on the set of numbers $\{1, 2, 3, \dots, p-2, p-1\}$. At the same time, you can calculate their quantity based on the number $p-1$ using the Euler's [1] function. Currently, there are no methods for obtaining information about their properties, but the recursive Fermat [1] function and modern computer technology make it technologically simple to calculate their values. However, if the prime number is large, then significant computational difficulties arise. The mathematical problems of the complexity of such calculations are presented in the monograph of the famous mathematician H. Rogers [2]. It is safe to say that without modern computer technology, the problem of calculating the set of all primitive roots cannot be solved in detail. First of all, it is necessary to take into account the fact that the process of calculating primitive roots is associated with a number of technological problems. The monography [3] proves that there exists an infinite number of positive integers for which the conditions of Fermat's theorem hold. Such numbers are called Carmichael numbers. Technologically, such numbers can be used in the recursive Fermat function, but this does not create primitive roots. Hence, we can conclude that detailed information is needed on the distribution of primes as generators of primitive roots and the recursive sequences associated with them. If we divide the value of the recursive iterations of a prime number by its magnitude, we get a sequence of real numbers in the interval $(0,1)$ that are random with an unknown probability distribution law. The study of their probability distribution law is an important and complex problem in information theory [4]. It should be noted that Kulbak's book was not the first fundamental work on information theory. The first scientific work on information theory was done by C. Shannon [5,6] in 1948 and 1949. In the following years, he published scientific works that became the basis of modern information theory [6]. When solving any scientific problem, both

theoretical and applied, it is necessary to take into account the fundamental principles of information theory.

In the works of the English mathematician, a specialist in mathematical statistics, Fisher, mathematical substantiation of the basics of information theory was made by creating a measure of the amount of information in information transmission systems through communication channels under various conditions [7]. The modern mathematical theory of information was created by Kulbak and studied in detail, and the results are presented in a monograph [4]. Later, the theory of dynamic properties of information was described on the basis of entropy [7]. Algorithms and methods of modern information theory have become the fundamental basis of modern information technologies [8,9]. Modern information technologies are directly related to almost all the problems of modern number theory and dynamical systems. The justification for this statement follows from a very well-known and simple fact. The essence of the whole range of problems in number theory is related to the simple fact that there are an infinite number of primes, but at the same time their properties have been studied only for a set of finite size. Most of the fundamental problems of number theory are related to the study of mathematical problems whose formulation is oriented to their entire set. Artin's conjecture in its generalized form depends entirely on the set of all natural numbers and the set of all primes [10]. Papers [11.13] substantiate the complexity of mathematical problems.

Artin's conjecture in its original form cannot be fundamentally analytically solved due to the fact that the emphasis of the formulation is on the notion of the primitive root of primes as generating elements of a group F_p^* . In this case, attention is focused on the author's confidence that the constant $A(2)$ will take the same values for any value of $a=k$ other than plus or minus one and full square. In addition, the author, in collaboration with other mathematicians of the time, provided a rationale containing erroneous facts that led him and his followers to a number of mistakes that have remained undiscovered until now. The range of Artin's misconceptions includes the belief that can be correctly

calculated based on the following expression, that this constant is the same for any value of $a > 2$, that it does not depend on $\text{ind}(a, p) > 1$, that the connection with Dirichlet's theorem concerning arithmetic progression is fragmentary. At the same time, neither he nor other authors of works related to the Artin's conjecture used modern computer and information technologies to verify the correctness of the estimates of Artin's constants obtained exclusively by analytical methods, sometimes under rather strict assumptions, which include the statement that the hypothesis about the zeros of the zeta function in the Dedekind form is valid. We note that we do not in any way exclude the use of methods of analytical number theory, provided that any steps in this direction should be substantiated in detail. One of such directions is the study of the laws of distribution of primitive roots as a first step, and in the future, indices $\text{ind}(a, p) > 1$.

Artin's conjecture applies exclusively to the primitive roots, but, as was proved in [10,14], if one does not have information about the laws of distribution of indices $\text{ind}(a, p)$, then for $c(5,1,x)$ it is impossible to clearly prove why $c(5,1,x) > c(2,1,x)$. Similarly, it would be impossible to explain why $c(2,1,x) > c(8,1,x)$, $c(2,1,x) > c(27,1,x)$, $c(2,1,x) > c(125,1,x)$, ... A considerable number of other examples can be given, primarily concerning the prime roots, and in general, the indices of primes when evaluating Artin's constants, not only for individual natural numbers a but also for systems of such numbers. If we consider Artin's problem as a fundamental problem of number theory, then the question arises of universal methods and algorithms for solving it for any number a , other than plus or minus one, and which is not a square. In the works of Artin and other authors, all efforts were focused on obtaining estimates of Artin's constants [4,5] using analytical methods.

The authors focused exclusively on creating a universal analytical [4.5] method for solving this problem. With the exception of the work of Bilhart's [6], in which computational methods and algorithms were practically not used. There was a simple but critical obstacle for this, which is related to the high computational complexity of the methods and algorithms for solving this problem. Fundamental works of Ch. Hooley [17,18] became the first step in

creating an analytical method for solving the problem. Works related to various forms of generalization of Artin's conjecture [7] and its transfer to algebraic number theory appeared, works related to the theory of simple ideals were published [1,2]. At the same time, Artin's conjecture remained at the same level of analysis of the methods of its solution as in the first fundamental works of a number of authors. The main problem was that all efforts were related to the development of methods for estimating Artin's constant in a quantitative, exclusively analytical form. Properties of prime numbers for which the number a is a primitive root, were not taken into account. In works [3,7,10] sets of consecutive prime numbers first became the basis of research methods for the formation of classes of prime numbers that correspond to certain values of the indices $\text{ind}(a, p)$. At the same time, the properties of prime numbers of certain classes of their sets were taken into account. The basis of such sets were their primitive roots and indices, their laws of distribution. The basis of such sets were their primitive roots and indices, their laws of distribution. Sets of consecutive prime numbers $[p_{n^*}, p_{n^*+M}]$, where n is the number of the smallest prime number, and M is their number in the given set became the basis for analyzing the processes of forming classes of primes with corresponding index values.

The basis of the technology of computer modeling of the processes of formation of classes of prime numbers and the calculation of generalized Artin's constants for any number a different from plus or minus one on the set of prime numbers was the method of statistical evaluation. The basis of this method was the hypothesis that any set of consecutive prime numbers of a certain power $[p_{n^*}, p_{n^*+M}]$ contains the necessary information for forming classes of prime numbers for evaluating all necessary indices. In the work [3] it was proved that if we use the recursive function of Fermat's little theorem under the condition that M ensures the statistical sufficiency of the sample of prime numbers, then we will always get the complete information necessary for estimating the Artin's constants for classes of prime numbers. According to the hypothesis, it is believed that the evaluation efficiency does not depend on the choice of a prime number p_n

provided that M ensures the efficiency of the statistical evaluation.

A problematic element of this method is the chosen procedure for selecting a set of prime numbers under the condition of sufficient statistics, if $M=500000$ and does not depend on the choice of P_n . Index values were calculated, which became the object of further analysis by the system of created algorithms for calculating indices and forming classes of prime numbers based on index values. It was proved that first of all it is necessary to take into account the primitive roots of prime numbers in a given set of prime numbers and to have information about their distribution laws. Until now, the laws of the distribution of primitive roots of prime numbers in the general case have not been studied. An even more difficult problem is the study of the laws of index distribution $ind(a, p) > 1$.

As already mentioned, the classes of prime numbers are associated with systems of arithmetic progressions, which in turn are associated with the Elliott-Halberstam hypothesis [11,12], according to which the structures of arithmetic progressions should be consistent in terms of the values of their differences. This is explained by the fact that when a set of consecutive prime numbers $[P_n, P_{n+M}]$ is chosen and the value $a = P$ is sufficiently large relative to P_{n+M} , it is necessary to significantly increase M , because otherwise the estimates of $\alpha(a, i, x)$ and properties of $R(a, i, x)$ may be incorrect with from the point of view of the properties of arithmetic progressions.

Analysis of the reduced sets of prime numbers gives rise to another problem. Each prime number p from this set is characterized by the value $\omega(p-1)$, which can be considered as a random number with a lognormal distribution law [13]. In this work, it is proved that when the prime number P_n increases, the mathematical expectation and variance also increase, which means that the structure of the set of consecutive prime numbers changes significantly if the increase exceeds a certain limit. It is possible that, starting from certain values of P_n , the reduced sets of consecutive prime numbers at a given value of M may change the properties of the set with respect to the estimated Artin constants.

In this case, it is possible that it will be necessary to increase M .

Analysis of the main conditions for the fulfillment of Artin's conjecture

Artin's conjecture belongs to a class of mathematical problems that is associated with a whole range of fundamental research in modern mathematics, which is related to the methods of its investigation and generalization [1,3]. The study of the decomposition of algebraic numbers into bases of simple ideals will be possible only if methods for solving the Artin's problem are developed, taking into account the entire system of problems that arise in its fundamental analysis. Let us consider Artin's conjecture as it was formulated by him in 1927 and discussed by him and Hasse [12,13]. The subject of analysis was the number 2 and the set of all primes P . According to their preliminary analysis, they proposed a formula for calculating the constant $A(2)$, which determines the density of the distribution of primes for which 2 is the primitive root. At the same time, a formula was proposed for calculating the numerical value $A(2)$, which has the following form:

$$A(2) = \prod_{q \in P} (1 - 1/(q(q-1)))$$

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The product is performed over the entire set of prime numbers $q \in P$. The rationale for this form of approximation of the density of primes $q \in P$ for which $a=2$ is the primitive root, is based on the following reflections of Artin and his partner in the analysis. From Artin's point of view, the event A can be interpreted as the product of independent random events $A_1, A_2, A_3, \dots, A_k, \dots, A_i$ and $A(2) = \prod A_i$. Each random event A_i has an opposite event $\overline{A_i}$ such that $P(A_i) + P(\overline{A_i}) = 1$. At the same time $P(A_i) = 1 - P(\overline{A_i})$, and always $P(\overline{A_i}) = 1/(q(q-1))$, equality is true because this event means that $p-1$ is divisible by q and at the same time $2^{(p-1)/q} \equiv 1 \pmod{p}$. Thus, each factor is an estimate of the probability of the next event. The number $1/(q(q-1))$ is the probability

that the following conditions are met for a prime number p . The number $p-1$ is divisible by a prime number q and the next equation is valid:

$$2^{\frac{(p-1)}{q}} \equiv 1 \pmod{p}, \quad 22 \backslash * \text{MERGEFORMAT} ()$$

which means that 2 is not a primitive root for a prime number p . It should be noted that, according to Dirichlet's theorem[1], for primes p , the probability that $p-1$ is divisible by q is $1/(q-1)$. The validity of Dirichlet's theorem is undeniable, and it has been verified both theoretically and by computer modeling. At the same time, agree that the probability that 2 is not a primitive root for a prime number p according to Artin is $1/q$ if the equality $2^{(p-1)/q} \equiv 1 \pmod{p}$ is valid, is not true in general. We will prove that this is an incorrect estimate. There are several reasons for its occurrence.

First of all, it is necessary to take into account that the equality $2^{(p-1)/q} \equiv 1 \pmod{p}$ is possible and m is the product of primes $m = q_1^{i_1} \dots q_k^{i_k}$ in certain powers. It is obvious that each q_i is divided by $p-1$. It follows that in this case the following set of inequalities is simultaneously true:

$$2^{\frac{(p-1)}{q_1}} \equiv 1 \pmod{p}, \dots, 2^{\frac{(p-1)}{q_k}} \equiv 1 \pmod{p} \quad 33 \backslash * \text{MERGEFORMAT} ()$$

It becomes necessary to choose one of the equations, but this choice is almost impossible to justify in terms of Artin's hypothetical strategy. It is difficult to estimate the probability of such cases for different values of a , but based on the results of computer modeling [15,16], in general, it is significantly different from zero. It can be argued that Artin's rationale, taking into account such cases, contradicts the actual methods of calculation of $P(a, i, x)$ and $\text{ind}(a, p)$. Subsequently, the value of the index $i = \text{ind}(a, p)$ is used to build the set $P(a, i, x)$, which includes the set of all primes $x=p$, for which $i = \text{ind}(a, p)$. Prime numbers are selected sequentially from the set of

primes $[P_n, P_{n+M}]$ where P_n - is the smallest prime number with the number n and P_{n+M} is the largest prime number with number $n+M$ in the set of all primes. To build a system of such sets of primes, a database of primes was used. Sets of cases with different index values $\text{ind}(a, p)$ is not taken into account by Artin in his estimates of the probabilities of different analysis options.

Let's consider the case when $a=5$ analysis of $P(5, i, x)$ classes for odd values i is not taken into account when the probability of $1/q$ is estimated incorrectly by the author. In [15] it was proved that at $a=5$ when the index i takes the value $(2k+1)5$ for every $k=0, 1, 2, \dots$, then the classes of primes $P(5, (2k+1)5, x) = \emptyset$ - are empty, and therefore $1/(q-1)$ incorrectly estimates the probability that $a=5$ is not an primitive root for a prime number p . This case is typical for all primes of the type $a=p=4k+1$. This leads to a major mistake in the assessment $1/(q-1)$, and therefore the probability of the opposite event is estimated as $1 - 1/(q-1)$ is a major mistake. By taking the product of all primes, in such cases we get the erroneous formula proposed by the author. The calculation of the product is even less justified. Such cases occur under different conditions, and therefore the theory of class formation $P(a, i, x)$ and estimation of Artin's constants $c(a, i, x)$ for all values a and i should be developed on the basis of fundamental information technologies.

In 1967, the work of K. Holey [17] appeared, in which the possibility of proving the correctness of Artin's conjecture by strictly mathematical methods based on the sieve theory was first considered. The next deepening of his results was his monograph [18], in which, based on the sieve theory and the assumption that GRH in the Dedekind form is valid, a proof of the validity of the Artin's conjecture for $a=2$ was given. The analysis of the Artin's problem in a consistent form is given in [15,16]. Estimates of the Artin's constant were obtained using numerical methods and computer modeling of the processes of forming classes of primes $P(a, i, x)$ and estimation of constants $c(a, i, x)$ for different values a and i , and a justification for their correctness is given. Note that Bilharz

[14] correctly estimated, without justification, that the Artin's constant $A(5)$ is five percent larger than in the case of $a=2$, but the possible reasons for this increase in values were not explained. The analysis of this case stimulated a more detailed study of mathematical properties in the Artin's problem when $a=4k+1$ and is a prime number.

In [15,16], an estimate of $A(5, i, x) = A(5)$ is given in the set of estimates of the values of the Artin's constants for the values of a from 2 to 32. The result $A(5) = 0.396 > A(2) = 0.3739$ is obtained by the methods of formation processes of classes $P(a, i, x)$ for $a=5$ and all values of the index i , and estimation of the constants $A(a, i, x)$ confirm the correctness of Bilharz estimate. It is a misconception that for all values of a , other than plus or minus one and the square, the Artin's constant must take the same value, which is still a significant misconception today. In the work [15], it was also proved that at $a=8$ or at $a=27$ and other cubes of primes, the Artin's constant takes the value 0.224. It should be noted that the range of values of the Artin's constants is much wider and their fluctuations are of much more fundamental importance. In the next section, **theorem (3)** gives a simple version of the proof of the reasons that determine this value of the constant at $a=5$.

Artin's conjecture became a factor that led to the formation of a whole range of research [1,3,15,16] in number theory, recursive function theory, mathematical theory of dynamical systems, and methods of encoding and information security. The core of the system of large-scale research was the theory of numbers, starting with natural and prime numbers. The next deepening of the research to the level of algebraic numbers with the use of various forms of their representation was the creation of fundamentally new methods for studying complex problems of modern mathematics based on analytical, algebraic, and computer technologies for solving them. Such processes have determined the need to create systematic combined techniques for solving problems of number theory, which are related, on the one hand, to the theory with deepening of analytical and algebraic methods, and, on the other

hand, to the development of computer methods for solving problems of modern mathematics.

The basis of the combined systematic method of analysis and deepening of methods for solving problems related to Artin's conjecture was a series of papers [15,16]. In these scientific works, it is proved that for a detailed study of Artin's conjecture, it is necessary to use in parallel the methods of algebraic, analytical number theory and methods of computer modeling of the processes of forming classes of prime numbers. The analysis will use arbitrary values of the index $\text{ind}(a, p)$, where a is any integer greater than one and p is a prime number, then the next equality always holds:

$$p-1 = \text{card}(a, p) \text{ind}(a, p), \quad 44 \backslash * \text{MERGEFORMAT}$$

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In [15,16] it was proved that every number a other than plus or minus one is a classifier of the set of all primes, and the set of all primes P is divided into a system of sets that have no numbers in common according to the following rule:

$$P(a, i, x) = \{p \mid \text{ind}(a, p) = i \ \& \ p = x \in \{p_{ix}, p_{ix+M}\}\} \ \text{and} \ P(x) = \bigcup P(a, i, x) \quad 55 \backslash *$$

MERGEFORMAT ()Numbers that are squares cannot be primitive roots, but they can create a classification of primes in which classes have only even indices, which will not be considered in this paper. It is necessary to pay attention to the variable x in all sets of primes $P(a, i, x)$ and the estimates of the Artin's constants $c(a, i, x)$, which indicates that these estimates are obtained on the set of primes $x=p$ belonging to the set of consecutive primes $[p_{ix}, p_{ix+M}]$. In cases where estimates obtained on the set of all primes will be considered, the x symbol will be absent. In these works, it is proved that qualitative methods for solving the Artin's conjecture and their application to the calculation of generalized Artin's constants require, first of all, knowledge of the laws of distribution of the primitive roots of any prime number p , taking into account the properties of $p-1$. The necessity to fulfill this requirement is due to a number of important factors. First of all, due to the fact that it has been proven that at different values of a and the index $i=1$ the constants $c(a, 1, x)$ take

different values, it became necessary to study in detail how the value of a and its properties affect the value of the constant $c(a, 1, x)$. An even more complicated problem is related to the study of which properties of a the value of the constant depends on. At the same time, it was proved that to obtain estimates of the constants $c(a, 1, x)$ it is necessary to take into account the properties of various primes that are part of the set of consecutive primes $[P_n, P_{n+M}]$ where P_n is the smallest prime number with the number n in the sequence of all primes, and there are M such primes in the sequence. To obtain such information at this stage, it was decided to limit ourselves to the primitive roots only, and therefore it became necessary to calculate the primitive roots for primes and to study the laws of their distribution and take into account their features in the given sets of consecutive primes. In the future, it is necessary to take into account the laws of distribution of indices of any value.

An extension of Artin's conjecture is that the classes of primes $P(a, i, x)$ have a deep connection with Dirichlet's theorem concerning arithmetic progression. The number a can be related to the module m as follows. Let us consider in detail the term "generalized and extended Artin's hypothesis", which means that for all $a \neq -1$ and $i = \text{ind}(a, p)$ the object of study is the properties of the sets $P_{x \rightarrow \infty}(a, i, x)$. Each such set is associated with an arithmetic progression. First of all, each set $P(a, i, x)$ can be described as follows:

$$P(a, i, x) = \{p \mid \text{ind}(a, p) = i\} \text{ where } x = p.$$

The number a can only be associated with a module m only in the following way. We assign $m=4a$ if a modulo 4 is not equal to one and $m=2a$ if $a=4k+1$. The justification of this rule is related to the theory of modules in number theory [1,20,21]. In general, the justification of such a rule is related to the theory of modules in number theory, and therefore we can consider the minimum values of the module at which certain properties of arithmetic progressions are preserved. In this case, it is important that the arithmetic progressions for classes of primes with even and odd indices do not have common elements. With the specified choice of module m , this requirement

is fulfilled. In the following papers, the problem of using module theory will be the subject of a separate study of the choice of m with certain properties of arithmetic progressions. In this way, we find a system of surpluses that are mutually prime with module m . In this paper, we present the fact that all classes of $P(a, i, x)$ are divided into a system of arithmetic progressions:

$$P(a, i, x | km + m_{1,j}) \text{ for } i=2k+1 \text{ and } m_{1,j} \in \{m_{1,1}, \dots, m_{1,q(N/2)}\} \quad 66 \setminus * \text{ ME}$$

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$$P(a, i, x | km + m_{2,j}) \text{ for } i=2k \text{ and } m_{2,j} \in \{m_{2,1}, \dots, m_{2,q(N/2)}\} \quad 77 \setminus * \text{ ME}$$

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For a correctly chosen module m for different values of a , a system of arithmetic progressions is always formed. At the same time, as a increases, the number of different progressions in the above scheme increases. The presented model of approximation of $P(a, i, x)$ classes by arithmetic progressions is the first step in deepening Artin's hypotheses. This means that the complexity of representing these sets and subclasses of primes $P(a, i, x | 4k + m_{1,j})$ and $P(a, i, x | km + m_{2,j})$ can vary significantly [21]. Note that at $a=5$ module $m=10$, and the complete reduced system of mutually prime excesses of 10 is the set $\{1, 3, 7, 9\}$. The classes of primes are divided into two subclasses with even and odd indices by the value of the index. It is proved that all classes of primes with odd indices modulo 10 have excess 3 and 7, and classes with even indices modulo 10 have excess 1 and 9, and only in this way. It can be argued that this is the simplest way to use the module. When moving to large values of a number a of complex form, the number of reduced remainders in the classes m_1 and m_2 increases significantly and the analysis of arithmetic progressions becomes a rather difficult problem.

It should be noted that if a is a complex positive integer, then the structures of classes of the cast types become completely unpredictable. For example, when $a=8$ and $a=27$ the Artin's constant is of the order $0.224\dots$, which means that the Artin's constants for different values of a can differ significantly. In this case, the extension of Artin's conjecture by describing the system of

arithmetic progressions modulo m classes with even indices $\{m_{2,1}, \dots, m_{2,\varphi(N/2)}\}$, the behavior of classes $P(a, 2i, x)$ differs significantly from classes with odd indices $\{m_{1,1}, \dots, m_{1,\varphi(N/2)}\}$ in terms of the values of the excesses, and the structures of the arithmetic progressions differ significantly, and therefore will be the subject of a separate paper.

Distribution of primes in sets of consecutive primes of finite size

The extension and generalization of Artin's conjecture on the estimation of constants $\alpha(a, i, x)$ in the general case is associated with the need to calculate their values also for values of $i = \text{ind}(a, p) > 1$, because at different values of a the value of the index $\text{ind}(a, p)$ is important for justifying the properties of the behavior of $\alpha(a, 1, x)$. For example, when a is a prime number and $a = 4k+1$, and $a=5$ it has long been known that $\alpha(5, 1, x) > \alpha(2, 1, x)$ [5, 9]. For all estimates of Artin's constants for $\alpha(5, 1, x)$ of the primes $x=p$ belonging to any set of consecutive primes $[p_k, p_{k+M}]$, provided that the following inequality also holds.. Above, the inequality was substantiated from the point of view of number theory, provided that the values of the constant were calculated on such sets for the given values of M . It is proved that even when $M=1000$, the estimates of the Artin's constants are correct for the first two decimal places, and the inequality is valid. Below, we will give a detailed analysis of the processes that determine such properties of a certain class of Artin's constants, provided that $a = 4k+1$.

With regard to the given set of consecutive primes, it is necessary to present the results of the analysis of their properties in terms of the relations between the primes of this set and the distribution of their primitive roots, taking into account the relations between them. Consider the representation of prime numbers p modulo 4. If $p=4k+1$ or $p=4k+3$, then you need to take into account that the distribution of the primitive roots of such numbers is significantly different. **Theorems 1 and 2** formulate elementary properties of the primitive roots of such primes. The choice of elementary properties can be an incentive to deepen the study.

It is easy to prove that if $p=4k+1$, then all its primitive roots have the following trivial properties.

Theorem 1. For any prime number $p=4k+1$ and a set $\{1, 2, 3, \dots, p-2, p-1\}$, the following holds:

1. if a number a is a primitive root of a prime number p , then $p-a$ is also its primitive root;
2. if a_1 and a_2 are the primitive roots of a prime number p , then $a_1 \cdot a_2$ cannot be a primitive root.

At the same time, the dynamics of the location of the primitive roots of the prime number $p=4k+3$ is significantly different.

Theorem 2. For any prime number $p=4k+3$ and set $\{1, 2, 3, \dots, p-2, p-1\}$, the following holds:

1. all the primitive roots a are located asymmetrically relative to the middle of the set;
2. if a_1 and a_2 are primitive roots of a prime number p , then $a_1 \cdot a_2$ cannot be the primitive root p .

These elementary theorems show that, in a given set of consecutive primes, these primes have different effects on the total distribution of the primitive roots, and for any set $[p_n, p_n+M]$ consecutive primes, the values of the Artin's constants are preserved with an accuracy determined by the value of M . The fact that points (2) coincide does not mean that the dynamics of primes of these classes of primes are not radically different.

None of the above trivial theorems has a fundamental impact on the processes of forming important information about the laws of distribution of primitive roots on the entire set of consecutive primes, regardless of the choice of p_n , but provided that M is at least a certain value. It should be noted that there is a rather profound difference between primes of types $p=4k+1$ and $p=4k+3$, which has been the subject of research for a long time in terms of their distribution in the set of all primes. It has now been proven that these classes of primes are equally probable [1,3,21].

The distribution of prime numbers $P_1 = \{p \mid p=4k+1 \mid k \in \mathbb{N}\}$ and $P_3 = \{p \mid p=4k+3 \mid k \in \mathbb{N}\}$ such that at the same time $|P_1|/|P| = |P_3|/|P| = 1/2$ and patterns of fluctuation distribution $p=4k+1$ and $p=4k+3$ on the set of all primes P have not been fully studied, but we can assume that the equality has a boundary character in the form:

$$\lim_{x \rightarrow \infty} |P_1(x)|/|P(x)| = \lim_{x \rightarrow \infty} |P_3(x)|/|P(x)| = 1/2$$

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Note that on sets of consecutive primes $[pn, pn+M]$. This type of relationship may deviate significantly from the $1/2$. The nature of this type of deviation in the probabilistic model for solving the "generalized Artin's conjecture" of this type of fluctuation must be taken into account when changing the values of n and M in $[pn, pn+M]$. Now it is necessary to analyze the reasons that lead to the need to analyze the distribution of not only the primitive roots, but also the indices $\text{ind}(a, p)$. Above, attention was drawn to the inequality $\alpha(5,1) > \alpha(2,1)$. Overall, more general inequality $\alpha(4k+1,1) > \alpha(2,1)$ holds. The reason for its appearance is the following theorem:

Theorem 3. For all $a=4k+1$ it is fair that $\alpha(4k+1, (4k+1)2l + (4k+1), x) = 0$ for all $l \geq 0$, which means that all the indices calculated by the above formula are equal to zero, and the next equation is always satisfied:

$$\sum \alpha(a, 2m+1, x) = 1/2$$

The proof of the theorem is based on a simple property of generalized Artin's constants with even and odd indices. It is easy to prove that for any a equality holds:

$$\sum \alpha(a, 2m+1, x) = \sum \alpha(a, 2m, x) = 1/2$$

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If some set of odd indices greater than one is zero, provided that always $\alpha(a,i) > \alpha(a,j)$, if $i > j$, then the sum will be equal to $1/2$ only if $\alpha(a,1) > \alpha(b,1)$, if $p=4k+1$ and b is a number of the type $4k+3$. Proofs of the existence of such indices, at which the corresponding constants are zero, are given in [15,16].

Let us thus consider the cases when there are a significant number of values of a for which the estimates of the Artin's constants range from 0,224... to 0.396... This process of oscillation is due to the processes of expansion and generalization of Artin's conjecture. It is essential for deepening our knowledge of number theory and recursion theory. A detailed analysis of the mathematical problems associated with Artin's conjecture shows that the structure of the classes $R(a, i, x)$ largely depend on the properties of a number a and prime numbers p when they interact in the process of computation using a recursive function are dynamic, the properties of which have not been studied when a and p interact in the process of computation. The lack of such information on the underlying processes of interaction between a and p , under conditions of compounding of a and large values of p , with significant compounding of p^{-1} , leads to practically unpredictable values of the results of calculations both by algebraic methods and in computer modeling processes. It is necessary to substantiate in more detail the parallel use of analytical methods and computer modeling technologies in the detailed analysis and solution of the extended and generalized Artin's conjecture. It is practically impossible to create a universal method for calculating Artin's constants for any a using analytical methods because it is extremely difficult to take into account the algebraic properties of the natural numbers a and the properties of the reduced recursive function on the set of all primes. In this case, it is necessary to take into account the properties of primes in terms of decomposition of p^{-1} into prime factors and to take into account the dynamics of changes in the probability law of distribution of primes by the value $\omega(p^{-1})$. The use of computer modeling methods for the formation of $R(a, i, x)$ classes and the calculation of Artin's constants $\alpha(a, i, x)$ is largely limited not only by computational complexity. At the same time, it is practically impossible to take into account all the problems that may arise in a full-scale analysis of a given mathematical problem. The prospect of the method of computer modeling of the processes of forming classes of prime numbers with the given a and index i is that in universal software systems it is possible only to change the

parameters if necessary and obtain the results of modeling. This strategy has been implemented and will be further used in this paper to create a system for analyzing the distribution of primitive roots.

A simple justification for the reason that $\alpha(5,1,x) > \alpha(2,1,x)$ on any interval $[p_n, p_{n+M}]$ of primes, as noted above, is due to the fact that the classes of $R(a,i,x)$ can be empty provided that $i > 1$ and a is not a square. It has been proven that $R(5,5,x) = \emptyset$ & $R(5,25,x) = \emptyset$ & ... and this is true for all $a = 4k+1$ and a is not necessarily a prime number. **Theorem 3** proves this fact in the general case. However, it is necessary to pay attention to one feature of this theorem. If $a = 4k+1$ is a prime number 53, then $\alpha(a,1,x)$ is practically the same as $\alpha(2,1,x)$ based on the same theorem, since only for index $i = 53$, $R(53,53,x) = \emptyset$ holds. The next index at which $R(a,i,x) = \emptyset$ is $i = 2 \cdot 53 + 53 = 159$, followed by the value of the index $i = 4 \cdot 53 + 53 = 265$ and so on. In his doctoral work, Ch. Ambrose [23] proposed the hypothesis that in an approximate form the valid relation $|R(a,i,x)| = 1/i^2 \cdot |R(x)|$ with a certain accuracy for any set of consecutive primes, provided that $i > 1$. It should be noted that this relation between classes of primes with a certain index value is performed in a rather approximate form separately for even and odd indices. It is easy to prove that $|R(53,53,x)| = |1/53^2 \cdot |R(x)||$ provided that the primes $x = p$ belong to the set $[p_n, p_{n+M}]$. This indicates that the fraction of such primes is insignificant, and therefore it cannot affect the value of $\alpha(53,1,x)$.

It should be noted that it is impossible to exclude the existence of other numbers a , whose behavior in this recursive procedure of forming classes of primes will not lead to similar effects, and this is the great difficulty of studying the extended and generalized Artin's conjecture. Such a complexity of this mathematical problem when trying to generalize it to the set of algebraic numbers based on simple ideals requires complete information about the laws of distribution of the primitive roots.

The value of the number a and its properties due to its decomposition into prime factors significantly affect the classes of primes and the corresponding generalized constants that arise

when it is necessary to study the structure of the classes $R(a, i, x)$ for all values of $\text{ind}(a, p)$ and to estimate the value of the constant $\alpha(a, i, x)$. For example, if $a=128$, then the question arises as to the value of $\alpha(128, 1, x)$. In Artin's time, the reduced value would have been the same as when $a=2$. It is not yet possible to estimate the value of $\alpha(a, i, x)$ for any values of a and i , even if it is roughly estimated, unless one has deeper knowledge. The analysis of the estimates of Artin's constants in a generalized form for any values a and index i by the methods of computer modeling of the processes of formation of classes of primes $R(a, i, x)$ on sets of consecutive primes of the type $[pn, pn+M]$ leads to the conclusion that the estimates depend on the value of M . Given the number $M=500000$ and any value of p_n , the estimates of the constants $\alpha(a, 1, x)$ coincide for four to five decimal places. When M is increased to 10^6 , the situation does not change. There is a complicated explanation for this fact. As M increases gradually and provided that the given sets of primes do not intersect, the structure of the sets of primes gradually changes with respect to the following system of properties. The set $[pn, pn+M]$ of primes is characterized by the distribution law $\omega(p-1)$. In [18] it is proved that the maximum value of $\omega(p-1)$ on the given intervals of consecutive primes increases when p_n increases by an amount of the order of 10^7 .

There are no effective algorithms and methods for estimating $\alpha(a, i, x)$ and constructing classes $R(a, i, x)$ for any values of a and i in the form of a mathematical method for which a theoretical justification has been created.

Consider the case when $a=p^3$ and p is a prime number. In this case, the next theorem holds:

Theorem 4. If $a=p^3$ and p is a prime number, then the Artin's constant $\alpha(a, 1, x)=0.224$ is constant for all values of a this type. The validity of the theorem follows from the analysis of the behavior of $\alpha(2, 1, x)$ for all primes p . Let p be any prime number for which 2 is its

primitive root. At the same time $2^{(p-1)} \equiv 1 \pmod{p}$. Consider the case when $p-1$ divisible by 3. Then the equality $2^{(p-1)} = (2^{(p-1)/3})^3 = (2^3)^{(p-1)/3} = 8^{(p-1)/3} \equiv 1 \pmod{p}$ is valid. This equality arises when a prime number p is such that the number 8 is not its primitive root and has index 3. There are an infinite number of such primes based on Dirichlet's theorem and the analysis of cases where 8 is not a primitive root. By the method of computer modeling of the processes of forming classes $A(8, i, x)$, it was proved that $\alpha(8, 1, x) = 0.224..$. The same transformations are possible for $a=3$ or $a=5$, or $a=7$ and so on. For $a=27$ and $a=125$ the same estimates of $\alpha(27, 1, x)$ and $\alpha(125, 1, x)$ were obtained. At the same time, estimates were obtained for values $i > 1$.

Properties of the distribution of classes of prime numbers by the value of the indices

Modern problems of number theory are largely related to the methods of sieve theory [1,2] and methods of their solution have been created. This observation to some extent refers to Artin's conjecture. However, in this problem, the application of the sieve method has a somewhat limited character, which is due to the stability of estimates of Artin's constants for any set of values of the number a on the set of all prime numbers. At this time, there is no complete information about the laws of distribution of prime numbers based on the value of $\omega(p-1)$. The Erdős-Katz theorem [26] regarding the normal distribution law of $\omega(n)$ probabilities for natural numbers cannot be applied for the following reasons. First, there is a fundamental difference between primes and natural numbers in terms of their distribution in the set of natural numbers. This is evidenced by works related to studies of the distribution of natural numbers with simple divisors of limited size [13]. In the cycle of such works, it was proved that the sieve method cannot be used to solve such problems. Secondly, when analyzing the laws of the distribution of prime numbers, it is not the value of the prime numbers that is used, but the value $\omega(p-1)$ and their properties are related to the properties of this quantity. In connection

ction with this simple remark, when applying the sieve method, it is necessary to take into account the problems that may arise.

Note that the "stability" with accuracy to ε^{p_n} of estimates for $\alpha(a, i, x)$ and properties of classes $R(a, i, 1)$ for all possible values of the index on the sets $[P_n, P_{n+M}]$ is determined by the distribution laws of $\text{ind}(a, p)$ for any P_n on set $\{1, 2, 3, 4, 5, \dots, k-1, k, k+1, \dots, P_n-2, P_n-1\}$. Indices can be even or odd, while their number is the same $(P_n-1)/2$. Until now, the distribution laws of both even and odd indices have not been studied in detail in the general case. Possessing knowledge about the properties of such laws is important for creating a detailed model for solving problems of Artin's conjecture. The law of distribution of odd indices is still unknown, and even more intriguing is the absence of any data regarding the law of distribution of primitive roots. If you create methods of computer modeling of the processes of class formation $R(a, i, x)$ and evaluation of $\alpha(a, i, x)$ based on analytical methods, then information about the laws of distribution $\text{ind}(a, p)$ on reduced ordered sets of prime numbers is simply necessary.

The first serious step in proving the correctness of Artin's conjecture was made in the works of Ch. Hooley [17,18]. In these works, attention was focused exclusively on the case when $a=2$. The author in [17] gave a simple justification of the method he created in the following form. A simple criterion from index theory allows us to distinguish between prime numbers p for which 2 is a primitive root modulo p . The theory of indices shows that for $p \neq 2$ and any prime divisor q of $p-1$, the equation $\mu_1^q \equiv 2 \pmod{p}$ is valid. Note that μ is a certain number that can be calculated based on the recursive function $X(0)=1, X(1)=a, X(n+1) \equiv aX(n) \pmod{p}$ (12), which is the basis of Fermat's little theorem [1,19]. We obtain the number 2 for a certain power in the process of recursive calculations for any primitive root of a given prime number p . Note that when we replace the primitive root of a given prime number with another primitive root, we get a different value of μ and the number 2 is obtained as a residue modulo p .

The transition to another primitive root leads to the fact that we will get the residue 2 on another iteration, which means that the index, which is the degree in the recursion, again according to the theo

ry of indices in this case is divided by a prime number q that divides $p-1$ and in the general case may not coincide with q and the equation $\mu_2^{q_2} \equiv 2 \pmod{p}$ is valid. Thus, we get a set of equations that in general do not allow us to come to the conclusion reached by the author. This case is due to the fact that in this case $\text{ind}(2, p) = m$, where m is a composite number. Such cases can occur frequently. In this analysis, the main role belongs to the process of recursive calculations, because only this process allows us to uniquely calculate the value of the recursion cycle $\text{card}(2, p)$. The value of the index can be calculated uniquely only using the expression:

$$\text{ind}(2, p) = (p-1) / \text{card}(2, p)$$

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Note that in the statement of Ch. Hooley it should be about this value $\text{ind}(2, p)$ and no other. The probability of such a case is different from zero, and therefore the lack of its consideration is an inaccuracy in his conclusion. We can assume that for the number $a=2$ such inaccuracy is insignificant.

It should be noted that in this case the prime number p has many primitive roots. Based on a rather complicated analysis, an analytical estimate for $A(2)$ of a rather complex form was obtained.

In the monograph [15] the estimation of $A(2)$ was given in a reasonable form. The author drew attention to the simple fact given above without analyzing the given variants from the theory of prime number indices. In his opinion, $a=2$ is not a primitive root of a prime number p if q divides $p-1$ and, according to the theory of indices, when calculating for any primitive root, we will find the number j in the recursion of Fermat's small theorem, that the equality will be true:

$$j^q \equiv 2 \pmod{p}.$$

This fact became the basis for proving the mathematical form of Artin's constant estimation on the basis of a rather complicated mathematical method, assuming that Riemann's hypothesis in Dedekind's form is valid. In general, for arbitrary values of a different from

2 this method is difficult to adapt to new conditions and it will not always be correct due to the fact that when estimating Artin's constants, under certain conditions, it is necessary to take into account the influence on the estimation processes of the values of $\text{ind}(a, p) > 1$ and the properties of the corresponding classes $R(a, i, x)$ of primes.

For example, for $a=5$, as proved in the work [3], it is necessary to take into account the value of a certain system of odd indices. In the case when the number a for a given prime number p is not a primitive root, it is necessary to take into account the influence of the index system and the fact that the indices can be complex numbers. Let's assume that $\text{ind}(a, p) = m$. The next step is to analyze all variants of the value of m . If $m=q$ and q is a prime number, then the further analysis can coincide with the analysis of Ch. Hooley, and in the case when m is not a prime number, there are many options associated with its decomposition into prime factors. At the same time, a significant number of options arise due to the fact that in this case it is impossible to simply choose one of the factors into which $m = \prod q_i^{\alpha_i}$ is decomposed. Such a choice cannot be simply made without detailed justification. Thus, the sequence

$$a^{(p-1)/q_i} \equiv 1 \pmod{p} \text{ when } m = \prod q_i^{\alpha_i} > q_i \quad 1111 \backslash * \text{ MERG}$$

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cannot be clearly justified and arbitrary choices lead to errors. The above remark is another factor justifying the need to apply the method of computer modeling of the processes of formation of classes of $R(a, i, x)$ for all $\text{ind}(a, p) = 1$ and estimation of constants $c(a, i, x)$.

Let's consider the dynamics of the formation of classes of prime numbers and their structure at different values of the indices in ascending order in the set of basic, in a certain interpretation, values of the number $a \in \{2, 5, 8, 20, 27, 125\}$. The given values of the number a are conditionally called basic due to the fact that based on the number $a=2$ Artin substantiated his conjecture, and all other values are related to the maximum and minimum values of Artin's constants. Of course, this property has not yet been proven by mathematical methods. Based on our analysis, this is true. **Table 1** shows the powers of

the classes of primes for indices from one to twenty, calculated for the set of consecutive primes $[P, P_{5 \cdot 10^5}]$ based on the method of computer modeling of the processes of formation of classes of prime numbers using the recursive Fermat function.

Table1. Dynamics of Changes in the Structure of Classes of Prime Numbers

Index	$P_{2,i,x}$	$P_{5,i,x}$	$P_{8,i,x}$	$P_{20,i,x}$	$P_{27,i,x}$	$P_{125,i,x}$
1	187111	196980	112331	197025	112185	118235
2	140325	133019	84138	132924	112362	79878
3	33188	34968	99679	34954	99718	104950
4	23349	33156	14016	33172	0	19869
5	9419	0	5672	0	5653	0
6	24969	23639	74759	23551	49921	70806
7	4457	4700	2677	4718	2692	2755
8	17468	8385	10429	8227	0	5623
9	3675	3834	11050	3845	11047	11667
10	7144	14058	4430	14185	5591	8441
11	1650	1838	1025	1747	1067	1116
12	4131	5935	12460	5960	37323	17731
13	1219	1229	714	1270	700	734
14	3304	3100	239	3116	2670	1872
15	1684	0	5002	0	5642	0
16	4397	2145	2604	2059	0	1284
17	2733	736	418	725	441	491
18	2733	2634	8392	2601	4473	7969
19	551	579	333	612	313	364
20	1246	3570	742	3538	0	2152

If we divide all of the values by 500000, we get Artin's constants. The classes $P_{5,i,x}$ and $P_{20,i,x}$ correspond to the maximum values of Artin's constant, and the classes $P_{8,i,x}$, $P_{27,i,x}$ and $P_{125,i,x}$ correspond to the minimum values. A simple analysis of each class shows that the dynamics of their formation has a rather complex nature, the analysis of which by analytical methods in the general case, if possible, requires the creation of complex methods. In our opinion, it is almost impossible to create a universal analytical method. The dynamics of $P_{27,i,x}$ and $P_{125,i,x}$ is significantly different from $P_{8,i,x}$ despite the fact that the Artin's constant takes the same values.

It can be concluded that in the processes of forming classes of prime numbers in the generalized and extended Artin's conjecture, for its improvement and deepening, it is necessary to have information about the laws of distribution of primitive roots, and in general, the indices of prime numbers.

Properties of the distribution of primitive roots of prime numbers

Thus, the equation of the Riemann zeta function in the Dedekind form under such circumstances is difficult to reasonably apply in methods of solving the problem at different values of a as a classifier of the set of primes. Therefore, it is necessary to create the basis of methods for analyzing the distribution of primitive roots and indices for prime numbers. Consider the algorithm for calculating $\text{ind}(a, p)$ for all a from an ordered set of integers associated with a prime p :

$$\{1, 2, 3, 4, 5, \dots, p-1, p+1, \dots, p-2, p-1\} = A_p$$

Let us define $\text{ind}(a, p) = (p-1) / \text{card}(a, p)$ where $\text{card}(a, p)$ is equal to the smallest k at which $a^k \equiv 1 \pmod{p}$. Obviously, if $k = p-1$ then $a^{p-1} \equiv 1 \pmod{p}$ and $a^{(p-1)/n} \not\equiv 1 \pmod{p}$, for any prime divisor n of the number $p-1$. In the case when $k < p-1$ then k always divides $p-1$ and $\text{ind}(a, p) > 1$, and if a is a primitive root, then $\text{ind}(a, p) = 1$. Suppose that a is a primitive root of a prime number p . Consider the basics of index theory in recursion:

$$x(0) = 1, x(1) = a, x(n+1) \equiv ax(n) \pmod{p}$$

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and n changes up to $n = p-1 = \prod_{i=1}^k p_i^{\alpha_i}$. Let's calculate the Euler function $\varphi(p-1) = \prod_{i=1}^k p_i^{\alpha_i-1} (p_i-1)$. The given function determines the number of numbers that are mutually prime to $p-1$. In recursion (1) in $x(n)$, the value n will be called the index, and the $x(n)$ is the residue modulo p . It is obvious that if a is a primitive root of p the

index runs through all values from 1 to $p-1$. It is clear that between indices and residuals for each a , which is the primitive root of a given prime number p , there is a one-to-one correspondence with respect to $p-1$ and vice versa. Suppose that the set of all primitive roots is known:

$$\{a_1, a_2, a_3, \dots, a_{\varphi(p-1)}\} \quad 1313 \backslash * \quad \text{MERGEFORMAT}$$

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We will assume that if $a = a_1$, then $A(n)$ in (1) will run through all elements of the same set of values, but in a different sequence. In general, there is no simple algorithm for finding this sequence. Based on Euler's theorem of primitive roots, the following theorem holds.

Theorem 5. If in $A(n)$ index n is not divisible by any prime divisor $p-1 = \prod_{i=1}^k p_i^{\alpha_i}$, then residue $A(n) \equiv a_1$ is the primitive root of p [12]. The proof follows from Euler's theorem of primitive roots. It is obvious that in such cases n is a mutually prime number with and therefore in the recursion $A(n+1) \equiv aA(n) \pmod{p}$ such numbers, that is, the primitive roots will always be $\varphi(p-1)$. It follows that if in the sequence $A(n)$ n is divisible by the number k that divides $p-1$, then $A(n)$ is not a primitive root of p and this means that when $A(n) = a$ then $\text{ind}(a, p) > 1$. Thus, we find the largest value of k , at which this number divides n in $A(n)$. Then we assign $\text{ind}(a, p) = k$ and $\text{card}(a, p) = (p-1)/\text{ind}(a, p)$, and therefore $a^{(p-1)/k} \equiv 1 \pmod{p}$ is fair. Thus, the theorem is true.

Theorem 6. If in the sequence of $A(n)$ calculation is based on the primitive root and the index n is divisible by the maximum number k , then $\text{ind}(a, p) = k$ and $\text{card}(a, p) = (p-1)/k$. This theorem is true for any primitive root a_j of the set of all primitive roots $\{a_1, \dots, a_j, \dots, a_{\varphi(p-1)}\}$. Based on the above considerations, it is possible to formulate the theorem.

Theorem 7. In the set $\{1, 2, 3, \dots, p-1\}$ for each primitive root a_j in the Fermat recursion (12) for any possible divisor l of $p-1$ there is a set

of numbers $\{a_{l,1}, \dots, a_{l,((p-1)/l)}\}$ which are the remainders of the recursion modulo p whose indices are $\text{ind}(a_{l,j}, p) = l$ and $(p-1)/l = \text{card}(a_{l,j}, p)$, and in the corresponding index system each of them is divided by l and l is their greatest divisor.

The validity of the theorem follows from the fact that all numbers of the set $\{a_{l,1}, a_{l,2}, \dots, a_{l,((p-1)/l)}\}$ are mutually prime to $\varphi((p-1)/l)$ and are generating elements of the corresponding subgroups of the group of residues Fp^* .

Thus, we get the set of numbers a , for which $\text{ind}(a, p) = l$. It should be noted that in the above analysis it was assumed that the smallest primitive root was known. In the general case of choosing a set of consecutive primes $[p_n, p_{n+M}]$, the values p_n and M can be chosen to be large, and therefore the following calculations by computer modeling methods of the formation of classes $R(a, i, x)$ and the calculation of constants

$\alpha(a, i, x)$ can be significantly complicated due to the fact that for each subsequent prime number it is necessary to find the smallest primitive root. In the absence of such information, finding the smallest primitive root can significantly complicate the calculation [15].

Thus, it can be concluded that $p-1 = \prod_{i=1}^k p_i^{\alpha_i}$ actually defines a whole set of different divisors $\{l_1, l_2, \dots, l_v\}$. The values $l_i = 1, \dots, l_v = p-1$ such that each l_j , based on theorem 7, defines $\varphi((p-1)/l_j)$ and $\{a_{1,j}, a_{2,j}, \dots, a_{\varphi((p-1)/l_j), j}\}$ is the set of mutually prime numbers such that $\text{ind}(a_j, p) = l_j$ and $\text{card}(a_{l,j}, p) = (p-1)/l_j$. The law of distribution for each divisor l_j on the set $\{1, 2, \dots, p-1\}$ unknown in the set of all $\text{ind}(a, p)$ for each p is still an unsolved mathematical problem. It is still possible to assume that such a law exists for all prime numbers p , but for numbers $p=4k+1$ the law of distribution of indices will differ significantly from the law of distribution for prime numbers of the type $p=4k+3$.

It should be noted that $\text{ind}(a, p)$ are divided into two classes of values a . In one class we will include those a for which $\text{ind}(a, p) = 2l + 1$, $l \in \{0, 1, 2, \dots\}$, and those a for which $\text{ind}(a, p) = 2l$, $l \in \{0, 1, 2, \dots\}$ will be assigned to the second class. The case when $\text{ind}(a, p) = 1$ belongs to the class $\text{ind}(a, p) = 2l + 1$ at $l = 0$ is radically different from the case $\text{ind}(a, p) = 2l$. It is generally accepted that if $\text{ind}(a, p) = 2l$ then a is a quadratic residue, while if $\text{ind}(a, p) = 2l + 1$ then a is not a quadratic residue. The number of quadratic residues and nonquadratic residues is the same - $(p-1)/2$, but their distribution on the set $\{1, 2, \dots, p-1\}$ is quite different. Consider the case $\text{ind}(a, p) = 1$. Let us find the distribution of the primitive roots by the distance between them. It is easy to prove that the average distance between the primitive roots is defined by the expression $(p-1)/\varphi(p-1)$. The need to find such a law is primarily due to Artin's conjecture and the need to find numbers a , at which the structure of the class. The term: the distribution of primitive roots of a prime number, should be considered from different points of view, and therefore the possibility of using probabilistic methods should not be excluded. Suppose that given a prime number p from the set

$$[p_{1/2}, p_{n+1/2}]: A_p = \{1, 2, 3, \dots, n-1, n, n+1, \dots, p-2, p-1\}$$

A simple analysis of this set shows that at $a=1$ or $a=p-1$ then they cannot be the primitive roots based on the trivial reason: $\text{card}(1, p) = 1$ and $\text{card}(p-1, p) \equiv \equiv 2 \pmod{p}$ for any p , and therefore $\text{ind}(a, p) = p-1$ and $\text{ind}(p-1, p) = (p-1)/2$ and moreover, they are exclusive elements of a similar set for any other prime number. If we consider any other number a from this set, then it is quite difficult to estimate the probability that a is the primitive root of p , and its measure, with increasing value of p , grows exponentially.

Still, there are simple ideas for analyzing the laws of formation of such estimates. According to Euler's theory, the number of primitive roots of a prime number p is determined by the Euler function

$$\varphi(p-1) = \prod_{i=1}^{k_p} p_i^{\alpha_i-1} (p_i-1), \quad \text{so we can use the estimate}$$

$\mu(a) = a_1(p) = \varphi(p-1)/((p-3) - k_p)$ where k_p is the number of squares in A_p , $\{1, p-1\} \notin A_p$. The distribution of primitives should give an estimate of the value of the smallest primitive root, in the literature there are attempts to obtain such estimates [11], but this is a rather difficult problem.

Since for each p the number of primitive roots is equal to $\varphi(p-1)$, the average distance between the primitive roots is $(p-1)/\varphi(p-1)$, provided that we consider cases where the smallest primitive root is not large. One of such properties can be considered the laws of distribution of distances between adjacent primitive roots and the laws of distribution of sets of primitive roots on intervals of a certain size. Consider a universal program for calculating all primitive roots of prime numbers of types $4k+1$ and $4k+3$ with different values of $\omega(p-1)$ based on a universal algorithm for calculating their primitive roots. And for each prime number, the set of all primitive roots is partitioned into a system of ordered subsets, each of which includes primitive roots, the distance between which is sequentially $1, 2, 3, \dots, n, \dots$. At the same time, distribution laws for all types and classes of prime numbers were systematically studied. It was found that in all cases the mathematical form of the distribution law had the same form. Graphical analysis led to the conclusion that the distribution has an exponential form, the parameters of which depend on the properties of the corresponding prime numbers. It was proved that the distance l between the primitive roots has an exponential form of the probability distribution law:

$$f_l(l) = \lambda_q e^{-\lambda_q l} \quad (14)$$

where $\lambda_q = \varphi\left(\frac{p-1}{q}\right)$ and λ_p is constant approximation law distribution distance l between numbers with indexes q for $p=4k+1$ & $p=4k+3$.

Let's consider an arbitrarily chosen system of consecutive eight primes. The analysis of the prime numbers given for example shows that the formation of primitive roots and their distribution differ significantly even for consecutive prime numbers. The number of primi

tive roots varies widely enough, even though the difference between the given primes is insignificant. This elementary fact became the fundamental basis for the search of laws according to which the generalized Artin's constants take the same values over such sets of consecutive primes. If the value of M is fixed, provided that its value ensures the accuracy of the estimation, then it is important to find a mathematical justification for such stability of the obtained estimates [6,8]. In fact, the formation of classes $P(a, i, x)$ and constants $c(a, i, x)$ for all values $i = \text{ind}(a, p)$ can be interpreted as the process of designing a system of characteristics of prime numbers over the set $[P_{\pi}, P_{\pi+M}]$.

Table 2. Structure of the Prime Factorization of $p-1$

p	p <i>decomposition</i>	$p-1$	$p-1$ factorization	$\varphi(p-1)$
2423	$4 \cdot 605 + 3$	2422	$2 \cdot 7 \cdot 173$	1032
2437	$4 \cdot 609 + 1$	2436	$2^2 \cdot 3 \cdot 7 \cdot 29$	672
2441	$4 \cdot 610 + 1$	2440	$2^3 \cdot 5 \cdot 61$	960
2447	$4 \cdot 611 + 3$	2446	$2 \cdot 1223$	1222
2459	$4 \cdot 614 + 3$	2458	$2 \cdot 1229$	1228
2467	$4 \cdot 616 + 3$	2466	$2 \cdot 3^2 \cdot 137$	816
2473	$4 \cdot 618 + 1$	2472	$2^3 \cdot 3 \cdot 103$	816
2477	$4 \cdot 619 + 1$	2476	$2^2 \cdot 619$	1236

The quality of the estimates of Artin's constants does not depend on P_n , but largely depends on the magnitude of M [8]. The distance between consecutive primitive roots of any prime number is distributed according to the exponential law. Mathematical

justification of the exponential form of the probability law of distribution of the distance between the primitive roots of any prime numbers $p > 10^5$ is not simple.

The correctness of this hypothesis follows from the fact that the processes of formation of the primitive roots are Poisson [16]. This statement is true because the formation of $\text{ind}(a, p)$ occurs under the influence of a significant number of factors that are independent of each other and the influence of each of them is insignificant, which leads to the formation of prime numbers, and therefore the theorem is true.

Theorem 8. For any prime number p , the set of primitive roots has the exponential form of the law in terms of the distance between consecutive primitive roots between them, $f_p(x) = \lambda_p e^{-\lambda_p x}$ where $\lambda_p = \varphi(p-1)$ and λ is constant approximation distance between primitive root.

Note that the statistical estimation of the parameter λ_p will be correct only if the estimates of the values of all primitive roots of the prime number p are known, i.e., the set $\{a_1, a_2, \dots, a_{\varphi(p-1)}\}$. It can be argued that the "smoother" a prime number is, the smaller $\varphi(p-1)$ and, correspondingly, λ_p . Thus, if the measure of smoothness of a prime number p_k is greater than the measure of smoothness of a prime number p_{k+1} , then the inequality $\lambda_{p_k} < \lambda_{p_{k+1}}$ is true.

It is necessary to pay attention to one more important property of primitive roots, which is associated with the existence of internal cycles of primitive roots that arise in the processes of implementing a recursive function when moving to another primitive root of a chosen prime number p . It can be argued that the smoother the $p-1$ that λ_p less. Since the term "smooth number" still does not have a clear definition, in the number theory [12,13] the above results will be considered as an experimental fact that can be substantiated meaningfully, but a detailed mathematical analysis for the correct proof of all theorems related to the distribution of primitive roots and indices of any value, similar to the analysis given in the works of Erdős-Katz [26] and Kowalski [27], will be the subject of a

separate work. Probabilistic proof of the exponential distribution law of $f_p^{(i)}()$ will be the subject of a separate study, parallel to the study of the laws of distribution of not only the primitive roots, but also indices $\text{ind}(a, p) > 1$. This problem in number theory is quite complicated and there are no sufficiently deep results [13].

Deepening of the exponential law of distribution of distances between adjacent primitive roots is associated with another property of the distribution of primitive roots and indices, which is that for any set of consecutive primes $[P_M, P_{M+M}]$, provided that M is not less than a certain value (assumed $M=500000$), then there exists a number Q that for each prime number p from this interval, the set $[1, 2, 3, \dots, p-2, p-1]$ is covered by a system of such intervals, where each interval contains the same number of primitive roots. This statement can be transferred to indices up to a certain value, which depends on the value of M . Deeper analysis of the theory of such coverage will be considered as a uniform law of distribution of primitive roots.

Conclusions

It is proved that Artin's conjecture in its extended and generalized form is important in modern number theory when studying the influence of natural numbers on the classification of primes by index values and the estimation of generalized Artin's constants. The author proves that, in general, the Artin's constant can vary within a wide range. In general, the calculation of Artin's constants requires both analytical methods and computer modeling technologies. It is proved that there is a deep fundamental connection between the classes of primes based on the index theory and arithmetic progressions.

It is proved that the processes of formation of Artin's constants of any integer largely depend on the properties of not only the primitive roots of primes, but also on the properties of their indices for different primes. Neglecting the properties of indices generally leads to inaccuracies in their estimates. The justification of the necessity to have detailed information about the properties of classes of prime numbers with different values of indices of their elements and their structural properties is given. The exponential law of distribution of the distance between the primitive roots of

any prime number p is found as a basis for studying the laws of distribution of their indices.

Table 3. Distribution of the Distance Between the Primitive Roots

Distance	p=242 3	p=243 7	p=244 1	p=244 7	p=245 9	p=246 7	p=247 3	p=247 7
1	444	189	377	610	614	281	274	617
2	244	126	228	306	306	162	178	304
3	147	86	135	150	152	121	114	158
4	87	58	94	80	79	89	68	86
5	48	34	46	38	36	61	68	38
6	24	32	32	19	19	27	46	8
7	12	24	16	9	14	19	22	10
8	14	10	14	4	2	20	16	6
9	4	8	6	3	3	10	8	6
10	4	6	2	1	1	14	6	2
11	2	6	2	1	1	2	10	
12		2	6			5	2	
13		2				1	1	
14		2				1	2	
15		2				1		
16	1	2				1		

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